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SOME EXTENSIONS OF THE WORK OF  
PAPPUS AND STEINER ON  
TANGENT CIRCLES

BY

J. H. WEAVER

A THESIS

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# SOME EXTENSIONS OF THE WORK OF PAPPUS AND STEINER ON TANGENT CIRCLES.

By J. H. WEAVER, Ohio State University.

**Introduction.** The figure of three mutually tangent semicircles with their centers in the same straight line was known among the Greeks as the "Shoemaker's Knife" ( $\alpha\rho\beta\eta\lambda\omicron\varsigma$ ). A few of the properties of the figure are found in the works of Archimedes.<sup>1</sup> Others occur in the Collection of Pappus.<sup>2</sup> After the Greek period we find no work done on the problem until Steiner generalized the results of Pappus and added several others dealing with the perspective properties of the figure.<sup>3</sup> Later Sir Thomas Muir added a theorem giving formulæ for various sets of radii involved.<sup>4</sup> Habicht has discussed some of the properties of elliptic functions connected with the figure<sup>5</sup> while M. G. Fontené has generalized certain formulæ arising from sets of tangent circles.<sup>6</sup>

In the present paper formulæ for the radii of certain sets of circles are developed and used to build up several types of infinite series which may be summed geometrically. Then some general properties of tangents and normals to conics

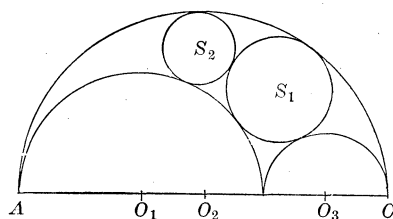


FIG. 1.

associated with three mutually tangent circles are set forth.<sup>7</sup> These properties lead to a quadrangular-quadrilateral configuration and incidentally furnish some methods for constructing conics. And finally some theorems connected with centers of perspectivity of the various sets of circles are proved.

**1. General Considerations.** Let there be two circles tangent internally, with centers  $O_1$  and  $O_2$  and let a circle with center  $S_n$  ( $n = 1, 2, \dots$ ) (Fig. 1) be tangent to these. Then  $S_n$  lies on an ellipse with foci  $O_1$  and  $O_2$ . If we take the midpoint of  $O_1O_2$  as origin and  $O_1O_2$  as the  $x$ -axis, the equation of the ellipse will be

$$\frac{4x^2}{(r_1 + r_2)^2} + \frac{y^2}{r_1 r_2} = 1, \quad (1)$$

where  $r_1$  and  $r_2$  ( $r_2 > r_1$ ) are the radii of the circles ( $O_1$ ) and ( $O_2$ ) respectively.

<sup>1</sup> *Works of Archimedes*, ed. Heath, Cambridge, 1897, Lemmas, 4-6.

<sup>2</sup> *Collectio*, ed. Hultsch., Berlin, 1876-8, Vol. I, pp. 209 and ff.

<sup>3</sup> Steiner, *Gesammelte Werke*, Berlin, 1881, Vol. 1, pp. 47-76.

<sup>4</sup> *Proceedings of Edinburgh Math. Soc.*, Vol. 3, p. 119. In the same volume, pages 2-11, J. S. Mackay has collected some of the simpler theorems connected with the problem.

<sup>5</sup> Konrad Habicht, *Die Steinerschen Kreisreihen*, Berne, 1904, 35 pp. In this work are found extensive references bearing on the subject.

<sup>6</sup> "Sur les cercles de Pappus," *Nouvelles Annales de Mathématiques* (4), tome 1918, pp. 383-90.

<sup>7</sup> The center of a circle tangent to two given circles lies on a conic having the centers of the two given circles as foci. This is, of course, equivalent to the definition that the sum or difference of the focal radii is constant. I have called such conics "associated" conics.

<sup>8</sup> In what follows circles will be designated by their centers in brackets.

Let  $\rho_n$  be the radius of  $(S_n)$  and let the coördinates of the point  $S_n$  be  $x_n$  and  $y_n$ . From a fundamental property of the ellipse we have

$$r_1 + \rho_n = a + ex_n, \quad (2)$$

where  $2a = r_1 + r_2$ , and

$$e = \frac{r_2 - r_1}{r_2 + r_1}.$$

Pappus has shown that if another circle  $(S_{n+1})$  with radius  $\rho_{n+1}$ , center at point  $x_{n+1}$ ,  $y_{n+1}$  and coming after  $(S_n)$  in the positive direction around the circles, is tangent to  $(S_n)$ , the following relation holds

$$\frac{y_n + 2\rho_n}{\rho_n} = \frac{y_{n+1}}{\rho_{n+1}}$$

or

$$\frac{y_n}{\rho_n} = \frac{y_{n-1}}{\rho_{n-1}} + 2 = \frac{y_1}{\rho_1} + 2(n-1). \quad (3)^1$$

**2. Formulæ arising from the figure of three mutually tangent circles with their centers in the same straight line.** Let there be three mutually tangent circles  $(O_1, O_2)$  and  $(O_3)$  having their centers in the same straight line and radii  $r_1, r_2$ , and  $r_3$  respectively (Fig. 1). Then let a series of circles  $(S_n)$  be drawn tangent to  $(O_1)$  and  $(O_2)$ , the first circle in the series being also tangent to  $(O_3)$  and each of the others tangent to the one preceding it in the series. There are two other such sets of tangent circles. The set tangent to  $(O_2)$  and  $(O_3)$  we will designate as  $(S'_n)$ , and the set tangent to  $(O_1)$  and  $(O_3)$  as  $(S''_n)$ . Let the radii of the various sets be  $\rho_n, \rho'_n$  and  $\rho''_n$  respectively, and the coördinates of the centers be  $x_n, y_n, x'_n, y'_n$  and  $x''_n, y''_n$  respectively. We will now consider the set  $(S_n)$ .

By means of equations (1), (2) and (3) and the use of induction we have in this particular case, since the  $y$ -coördinate of  $O_3 = 0$

$$\rho_n = \frac{r_1 r_2 r_3}{n^2 r_3^2 + r_1 r_3 + r_1^2}. \quad (4)$$

This result is arrived at by Muir and Fontené by different methods.<sup>2</sup> Also from equations (2) and (4)

$$x_{n-1} - x_n = \frac{(2n-1)r_3^2 r_1 r_2 (r_1 + r_2)}{[(n-1)^2 r_3^2 + r_1 r_3 + r_1^2][n^2 r_3^2 + r_1 r_3 + r_1^2]} = i_n, \text{ say.} \quad (5)$$

From the geometric properties of the figure

$$\sum_{n=1}^{\infty} \rho_n \quad (6)$$

is a convergent series, and if  $r_3$  approaches the limit 0, then (6) approaches the value  $\pi r_2/2$  but is 0 at the limit.

<sup>1</sup> Pappus, *Collectio*, p. 224.

<sup>2</sup> See Introduction.



Also

$$\sum_{n=1}^{\infty} i_n = 2r_1 + r_3.$$

If we define  $i_n$  as the  $n$ th intercept of the series  $(S_n)$  ( $i_1$ , projection of  $O_3S_1$  on  $AC$ ), then (4) and (5) are the formulæ for the  $n$ th radius and intercept in the series  $(S_n)$ . An interchange of  $r_1$  and  $r_3$  will give the corresponding formulæ for the series  $(S'_n)$ , while an interchange of  $r_2$  and  $r_3$  with  $r_2$  considered negative will give the corresponding formulæ for the set  $(S''_n)$ .

If  $r_2 = 2r_1$  we have the special case

$$\sum_{n=1}^{\infty} \rho_n'' = r_2/2.$$

We will now establish the following theorem.

**THEOREM:** If two circles  $(O_1)$  and  $(O_2)$  are tangent internally, and a circle  $(S)$  is drawn tangent to these two, such that  $SO_i$  ( $i = 1, 2$ ) is perpendicular to  $O_1O_2$  then it is possible to draw a circle  $(S')$  tangent to  $(O_1)$ ,  $(O_2)$  and  $(S)$  such that the four centers  $S, S', O_1$  and  $O_2$  determine a rectangle.

*Proof:* Let  $SO_2$  be perpendicular to  $O_1O_2$ , and let the coördinates of  $S$  and  $S'$  be  $x, y$  and  $x', y'$  respectively and let the radii be  $\rho$  and  $\rho'$ . Then

$$x = \frac{r_2 - r_1}{2}, \quad y = \frac{2r_1r_2}{r_1 + r_2}, \quad \rho = \frac{r_2(r_2 - r_1)}{r_1 + r_2}, \quad (6)$$

and by virtue of equations (1), (2) and (3)

$$y' = y \text{ and } x' = -x,$$

which proves the theorem.

**THEOREM:** If in the series  $(S_n)$ , the points  $S_n, S_{n+1}, O_1$  and  $O_2$  determine a rectangle, then  $r_1 = nr_3$ .

*Proof:* Equate the values of  $\rho$  given in equations (4) and (6).

Let the foot of the perpendicular from  $S_n$  to  $O_1O_2$  be  $P_n$ .

Let angle  $P_nS_nS_{n+1} = \angle B_n$ .

Then if  $r_1 = kr_3$  ( $k$  an integer or a rational fraction)

$$\tan B_n = \frac{(2k+1)(2n+1)}{2(n^2 + n - k - k^2)} \quad (7)$$

and the slopes of the lines of successive centers of the series  $(S_n)$  are all rational. Moreover if in (7)  $k$  is an integer,  $B_k = \pi/2$ , and

$$\sum_{n=k+1}^{\infty} (B_{n-1} - B_n) = B_k - \lim_{n \rightarrow \infty} B_n = \pi/2.$$

From the identity  $(\rho_n + \rho_{n+1})^2 = i_{n+1}^2 + (y_n - y_{n+1})^2$  we get by using equa-

tions (3), (4) and (5) the equation

$$\begin{aligned} [(2n^2 + 2n + 1)r_3^2 + 2r_1r_3 + 2r_1^2]^2 \\ = [(2n^2 + 2n)r_3^2 - 2r_1r_3 - 2r_1^2]^2 + [(2n + 1)r_3(r_2 + r_1)]^2 \end{aligned}$$

giving a triply infinite set of rational right triangles.

**3. Formulæ arising from three mutually tangent circles, one of which is tangent to the line of centers of the other two.** Let there be two circles ( $O_1$ ) and ( $O_2$ ) (Fig. 2) tangent internally at  $A$  and let a series of circles be drawn tangent to these, the first one in the series being tangent to the line  $O_1O_2$  and each of the others tangent to the one preceding it in the series. Let the radius of ( $O_1$ ) be  $r_1$  and of ( $O_2$ ) be  $r_2$ , and let the centers of the circles be  $S_n$ . Then from equations (1), (2) and (3), since  $y_1 = \rho_1$  we get by induction

$$\rho_n = \frac{4r_1r_2(r_2 - r_1)}{4(n^2 - n)(r_2 - r_1)^2 + (r_2 + r_1)^2} \quad (8)$$

$$i_n = \frac{32(n - 1)r_1r_2(r_2 - r_1)^2(r_2 + r_1)}{[4(n^2 - n)(r_2 - r_1)^2 + (r_2 + r_1)^2][(4n^2 - 12n + 8)(r_2 - r_1)^2 + (r_2 + r_1)^2]} \quad (9)$$

where for  $i_n$   $n \geq 2$ .

Here

$$\sum_{n=2}^{\infty} i_n = 2r_2 - i_1^1$$

and if  $r_1$  approaches  $r_2$

$$\sum_{n=2}^{\infty} \rho_n$$

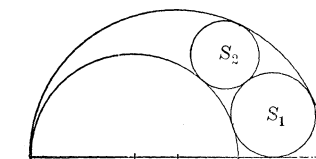


FIG. 2.

approaches  $\pi r_2/2$ .

**THEOREM:** If in this series the points  $S_n$ ,  $S_{n+1}$ ,  $O_1$  and  $O_2$  determine a rectangle then

$$r_2 = \frac{2n + 1}{2n - 1} \cdot r_1.$$

Let

$$r_2 = \frac{2k + 1}{2k - 1} \cdot r_1.$$

Then

$$\tan B_n = \frac{2nk}{n^2 - k^2}.$$

Moreover,

$$\sum_{n=k+1}^{\infty} (B_{n-1} - B_n) = B_k - \lim_{n \rightarrow \infty} B_n = \pi/2.$$

Also the equation  $(\rho_n + \rho_{n+1})^2 = (i_{n+1})^2 + (y_n - y_{n+1})^2$  gives the triply infinite set of rational right triangles<sup>2</sup>

$$[4n^2(r_2 - r_1)^2 + (r_2 + r_1)^2]^2 = [4n^2(r_2 - r_1)^2 - (r_2 + r_1)^2]^2 + [4n(r_2^2 - r_1^2)]^2.$$

<sup>1</sup> The distance from the point of contact of  $S_1$  with the diameter  $AO_2$  to the end of this diameter, on the side opposite from the point  $O_2$ , is taken as  $i_1$ .—Editor

<sup>2</sup> This result is but a special case of a rational right triangle with sides  $u^2 + v^2$ ,  $u^2 - v^2$ , and  $2uv$ .—Editor.

4. **Formulæ arising from a series of tangent circles, tangent to two given circles, the first circle in the series being tangent to a line tangent to the smaller circle and perpendicular to the line of centers.** Let there be two circles  $(O_1)$  and  $(O_2)$  tangent internally at  $A$  and let  $(O_1) < (O_2)$  and let the tangent to  $(O_1)$  perpendicular to  $O_1O_2$  be drawn and let a series of tangent circles be drawn tangent to  $(O_1)$  and  $(O_2)$ , the first circle in the series being also tangent to the perpendicular just drawn (Fig. 3).

Then from Pappus, Book IV., lemma XIX, we have

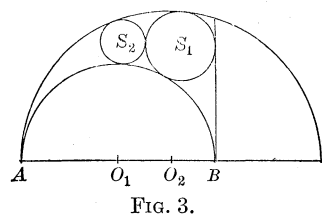


FIG. 3.

Let

$$\frac{r_2 - r_1}{r_2} = \frac{4\rho_1^2}{y_1^2}.$$

$$\frac{r_2 - r_1}{r_2} = m^2,$$

then

$$2\rho_1 = my_1. \quad (10)$$

Using equations (1), (2), (3) and (10) we have by induction

$$\rho_n = \frac{r_1 m^2}{(n-1)^2 m^4 + 2(n-1)m^3 + 1},$$

$$i_n = \frac{r_1 m^3 (2 - m^2) [(2n-3)m + 2]}{[(n-2)^2 m^4 + 2(n-2)m^3 + 1][(n-1)^2 m^4 + 2(n-1)m^3 + 1]}.$$

If  $r_1$  approaches  $r_2$ , the sum  $\sum_{n=1}^{\infty} \rho_n$  approaches  $r_2 \cdot \pi/2$  but is zero at the limit. Also  $\sum_{n=1}^{\infty} i_n = 2r_1$ , if  $i_1 = \rho_1$ . Let  $\angle B_n = \pi/2$ . Then since

$$\sin B_n = \frac{i_{n+1}}{\rho_n + \rho_{n+1}}$$

we have the relation

$$n = \frac{1 - m}{m^2},$$

and if  $n = 1$ ,  $r_1 : r_2 =$  side of decagon inscribed in a circle of unit radius. Here also as in sections (2) and (3) we may obtain an equation

$$\begin{aligned} & [(2n^2 - 2n + 1)m^4 + (4n - 2)m^3 + 2]^2 \\ &= [2n^2 - 2n)m^4 + (4n - 2)m^3 + 4m^2 - 2]^2 + [m(2 - m^2)((2n - 1)m + 2)]^2, \end{aligned}$$

which gives a doubly infinite set of rational right triangles.

5. **Formulæ arising from a series of tangent circles tangent to a given circle and a given straight line.** Let (Cf. figure 3) the series of circles  $(S_n)$  be tangent to the perpendicular at  $B$  and to  $O_2$ ,  $(S_1)$  being also tangent to  $O_1$ .

The centers  $S_n$  lie on a parabola with vertex  $O_1$  and focus  $O_2$ . Using  $O_1$  as origin the equation of the parabola is

$$y^2 = 4(r_2 - r_1)x. \quad (11)$$

Moreover

$$\rho_n = r_1 - x_n \quad (12)$$

and

$$(y_n - y_{n-1})^2 = (\rho_n + \rho_{n-1})^2 - (x_n - x_{n-1})^2. \quad (13)$$

By a substitution from (12) equation (13) reduces to

$$(y_n - y_{n-1})^2 = 4\rho_n\rho_{n-1}. \quad (14)$$

Let  $r_1 = \lambda r_2$ . Let  $D_n$  denote the sum of the odd-numbered terms in the expansion of  $(1 + \sqrt{\lambda})^n$ : and  $N_n$  the sum of the even-numbered terms, that is

$$D_n + N_n = (1 + \sqrt{\lambda})^n,$$

$$D_n - N_n = (1 - \sqrt{\lambda})^n.$$

Then using equations (11), (12) and (14) and induction

$$\begin{aligned} \rho_n &= \left(1 - \frac{N_n^2}{D_n^2}\right) r_1, \\ y_n &= \frac{2N_n}{D_n} \sqrt{r_1(r_2 - r_1)}. \end{aligned} \quad (15)$$

From (15)

$$\frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{(\sqrt{r_2} - \sqrt{r_1})^n}{(\sqrt{r_2} + \sqrt{r_1})^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{\sqrt{r_2} - \sqrt{r_1}}{2\sqrt{r_1}}.$$

Also we have

$$\sum_{n=1}^{\infty} (y_n - y_{n-1}) = 2\sqrt{r_1(r_2 - r_1)}.$$

## 6. Some properties of conics associated with three mutually tangent circles.

Let there be two circles  $(O_1)$  and  $(O_2)$  tangent internally at  $A$  (Fig. 4) and let the radii of these circles be  $r_1$  and  $r_2$  respectively, and let the radius of a circle tangent to these two and having its center on the line  $O_1O_2$  be  $r_3$  and let its center be  $O_3$ . Let  $O$  be the center of any circle tangent to  $(O_1)$  and  $(O_2)$  and let  $r$  be its radius.

The conic associated with  $(O)$  and  $(O_2)$  is an ellipse with the points  $O$  and  $O_2$  as foci, and passing through  $O_1$ : the conic associated with  $(O)$  and  $(O_1)$  is a hyperbola passing through  $O_2$  and having  $O$  and  $O_1$  as foci. Likewise we will have an ellipse passing through  $O$  and having  $O_1$  and  $O_2$  as foci. Draw from  $A$  the line  $AT$  tangent to the circles  $(O_1)$  and  $(O_2)$ . Then with the three circles  $(O_1)$ ,  $(O_2)$  and  $(O)$  the straight line  $AT$  there will be associated four parabolas<sup>1</sup> two of which pass through  $A$ , one through  $O_1$  and the fourth through  $O_2$ .

<sup>1</sup> See article "Some Properties of a Straight Line and Circle and their Associated Parabolas," *Annals of Math.*, second series, Vol. 19, pp. 174-5. Also "Some properties of circles and related conics," *Annals of Math.*, second series, vol. 20, pp. 279-280.



of one being on a line through  $O_3$  perpendicular to  $O_1O_2$  and the diameters of the other being segments of tangents to  $E_3$  cut off by the tangents at  $A$  and  $O_3$ .<sup>1</sup> It should also be noted in this connection that the point  $T$  is the pole of the line drawn from the point of tangency of any two of the circles to the center of the third circle with respect to the conic passing through that center.

Also if there is drawn at  $D$  a line perpendicular to  $O_1O_2$  and  $TC_1$  is produced intersecting this line in  $S$ , then  $N$ ,  $S$  and  $O_1$  are collinear.

**THEOREM:** If three circles are mutually tangent and tangents and normals be drawn to the three associated conics at the points of contact and the centers of the three circles, and if the normals of two of the conics be chosen, these will intersect by twos on a tangent to the third conic.

*Proof:* Consider the lines  $O_1N$ ,  $O_2N$ ,  $OO_1$ ,  $OO_2$ . These intersect in the points  $O$  and  $N$  which are on the tangent to  $E_3$ .

**THEOREM:** The axes and normals to two of the conics, together with the tangents to these two conics drawn at the centers of the circles determine two perspective triangles whose center of perspectivity is the intersection of the axis and normal to the third conic.

*Proof:* Consider the lines  $O_1N$ ,  $O_2N$ ,  $OO_1$ ,  $OO_2$ ,  $O_1T$ ,  $O_2T$ . These intersect by twos on the line  $TN$ . They may therefore be considered as the sides of two perspective triangles. Let the corresponding sides be

$$\begin{array}{ll} O_1T, & O_2T, \\ O_2N, & O_1N, \\ OO_1, & OO_2. \end{array}$$

These determine the perspective triangles  $A_{12}A_{13}A_{23}$  and  $B_{12}B_{13}B_{23}$  and the center of perspectivity is the point  $F$  on  $O_1O_2$  (see Fig. 4 where  $F$  is marked). But this point is also on the normal  $OB_{12}$ .

*Corollary:* There will be three such sets of perspective triangles and the centers of perspective will be collinear (second theorem before the last).

**THEOREM:**  $N$  and  $T$  are double points of an involution, of which  $O$  and the point ( $Y$ ), where  $TN$  intersects  $O_1O_2$ , are a conjugate pair, and therefore  $O$  and  $Y$  are inverse points with respect to the circle  $C_t$ .

The proof of this theorem follows immediately from a consideration of the quadrangle  $O_1A_{12}O_2B_{12}$ .

The following theorems are also evident.

**THEOREM:** If the three axes and the three normals to the three associated

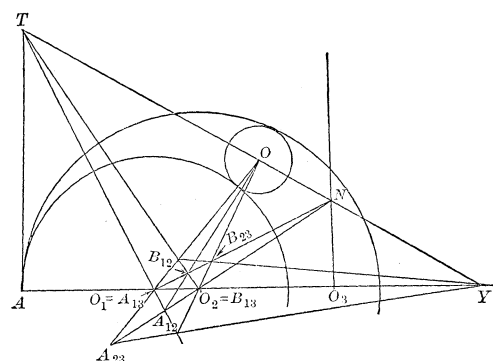


FIG. 5.

<sup>1</sup> See *Conics* of Apollonius, Book III, prop. 45 (Heath's ed., p. 114).

conics be drawn the axis and normal to each conic being taken as corresponding sides, they form two perspective triangles with  $T$  as center of perspective.

**THEOREM:** The four axes of perspective of the three circles and the six lines, three of which are normals and the other three are the tangents to the three conics at the three centers of the three circles, determine a quadrangular-quadrilateral configuration, whose diagonal triangle is the triangle determined by the three centers of the three circles.

In connection with the above discussion it should be noted that it furnishes a method for constructing points on a conic. For let  $O_1$ ,  $O_2$  and  $O_3$  be any three points on a line, and let the perpendicular be drawn at  $O_3$  and let  $N$  be any point in the perpendicular. Let  $O_1$  and  $O_2$  be points such that we have the order  $O_1O_2O_3$  or  $O_2O_1O_3$ . Then from  $N$  draw lines to  $O_1$  and  $O_2$ , making the angles  $\angle NO_1O_3$  and  $\angle NO_2O_3$ , and draw from  $O_1$  and  $O_2$  the lines  $O_1O$  and  $O_2O$  such that

$$\angle NO_1O_3 = \angle OO_1N,$$

$$\angle NO_2O_3 = \angle OO_2N.$$

Then the point  $O$  is on an ellipse. If we have the order  $O_1O_3O_2$ ,  $O$  will be on a hyperbola, and if  $O_1$  or  $O_2$  is at infinity we have a parabola. And in each instance  $O_1$  and  $O_2$  are foci of the conic. This also gives a method for establishing a (1, 1) correspondence between the points of a conic and the points of a straight line.

**7. Some Projective Properties of the Figure in Section 2.** Let  $A$  and  $C$  be the ends of the diameter  $O_1O_2$  of the circle ( $O_2$ ) (Fig. 1), and let there be drawn from  $A$  lines to  $S_n$  and from  $C$  lines to  $S_n'$ , and let  $C$  and  $C'$  be the angles that these lines make with  $AC$ . Then

$$\tan C = \frac{2nr_3}{r_1 + r_2}, \quad (16)$$

$$\tan C' = \frac{2nr_1}{r_2 + r_3}. \quad (17)$$

By means of equations (16) and (17) we may find the equations of the lines  $AS_n$  and  $CS_n'$ , a solution of which reveals the fact that the line  $AS_n$  and the line  $CS_n'$  intersect on a line through  $S_1$  perpendicular to  $AC$  in points whose ordinates are  $2\rho_n$ .<sup>1</sup>

By very simple analytical considerations we may prove the following

**THEOREM:** The triangles  $S_{n+1}S_nS_{n-1}$  and  $S_{n+1}'S_n'S_{n-1}'$  are perspective and their center of perspective is the external center of perspective of the circles  $O_1$  and  $O_3$ .

**THEOREM:** The locus of the point of contact of two tangent circles which are tangent to two given tangent circles (internally tangent) is a circle whose center is the center of perspective of the two given circles and whose radius is the harmonic mean between the radii of the two given circles.

<sup>1</sup> In this connection see Steiner, p. 69 and ff.

